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NORM ATTAINING OPERATORS AND RENORMINGS OF BANACH SPACES

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ABSTRACT

We define two geometric concepts of a Banach space, property α and β, which generalize in a certain way the geometric situation of l and c_0. These properties have been used by J. Lindenstrauss and J. Partington in the study of norm attaining operators. J. Partington has shown that every Banach space may $(3+\varepsilon)$-equivalently be renormed to have property β. We show that many Banach spaces (e.g., every WCG space) may $(3+\varepsilon)$-equivalently be renormed to have property α. However, an example due to S. Shelah shows that not every Banach space is isomorphic to a Banach space with property α.

Introduction

This work is entirely based on the remarkable paper of J. Partington [7], where it is shown that every Banach space may be equivalently renormed to have "property B" (cf. [5], for a definition see below). Actually, Partington shows that these renormings verify a criterion (called "property β" below), for which J. Lindenstrauss [5] showed that it implies property B.

The present author gives a "predual" version of Partington's construction which applies to a large class of Banach spaces and shows that these spaces have an equivalent renorming verifying "property A" (cf. [5]). In particular we show that many spaces failing RNP (e.g. c_0, $L^1(\mu)$, $C[0,1]$, l^∞) have a renorming verifying property A. Hence property A is not equivalent to the Bishop–Phelps property (which has been shown by J. Bourgain to be equivalent to RNP [1]).

1. Definitions and notations

The Banach spaces are assumed — only for simplicity — real. We say that an operator T from a Banach space X to a Banach space Y *attains its norm* if there is an $x \in X$, $\|x\| = 1$ such that $\|T\| = \|Tx\|$.

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DEFINITION 1.1. ([5]) (a) The Banach space X has *property A* if for every Banach space Y the norm attaining operators are dense in $L(X, Y)$.

(b) The Banach space Y has *property B* if for every Banach space X the norm attaining operators are dense in $L(X, Y)$.

J. Lindenstrauss [5] used the following two criteria for property A and B, which we call property α and β in view of Proposition 1.3.

DEFINITION 1.2. (a) The Banach space X has *property* α if there is a λ with $0 \leqq \lambda < 1$ and a family $\{(x_\alpha, y_\alpha)\}_{\alpha \in I} \in X \times X^*$, with $\|x_\alpha\| = \|y_\alpha\| = \langle x_\alpha, y_\alpha \rangle = 1$ such that

(i) for $\beta \neq \alpha$, $|\langle x_\alpha, y_\beta \rangle| \leqq \lambda$,

(ii) the unit ball of X is the closed, circled convex hull of $\{x_\alpha\}_{\alpha \in I}$.

(b) The Banach space Y has *property* β if there is a λ with $0 \leqq \lambda < 1$ and a family $\{(y_\alpha, x_\alpha)\}_{\alpha \in I}$ in $Y \times Y^*$, with $\|y_\alpha\| = \|x_\alpha\| = \langle y_\alpha, x_\alpha \rangle = 1$, such that

(i) for $\beta \neq \alpha$, $|\langle y_\alpha, x_\beta \rangle| \leqq \lambda$,

(ii) for $y \in Y$, $\|y\| = \sup\{|\langle y, x_\alpha \rangle| : \alpha \in I\}$.

A typical example for property α is l^1, while for property β the spaces c_0 or l^∞ are typical.

PROPOSITION 1.3. (a) *Property* α *implies property* A.

(b) *Property* β *implies property* B.

PROOF. As regards (b) this is proposition 3 of [5].

As regards (a) we also can easily reduce it to the results of J. Lindenstrauss, noting the following

FACT. Let X have property α. Then for every $\alpha \in I$, $\varepsilon > 0$ and $\|x\| \leqq 1$

$$\langle x, y_\alpha \rangle > 1 - \varepsilon(1 - \lambda) \Rightarrow \|x - x_\alpha\| < 2\varepsilon.$$

It follows that the unit ball of X is the closed circled convex hull of the uniformly strongly exposed family $\{x_\alpha\}_{\alpha \in I}$, hence proposition 1 of [5] implies that X has property A.

PROOF OF THE FACT. We may suppose

$$x = \sum_{i=1}^{n} \mu_i x_{\alpha_i}$$

where $\sum_{i=1}^{n} |\mu_i| \leqq 1$ and $\alpha_1, \cdots, \alpha_n \in I$. We also may assume $\alpha_1 = \alpha$. Then

$$1-\varepsilon(1-\lambda)<\sum_{i=1}^{n}\mu_i\langle x_{\alpha_i},y_\alpha\rangle$$

$$\leqq \mu_1\langle x_\alpha, y_\alpha\rangle+\sum_{i-2}^{n}(|\mu_i|\cdot|\langle x_{\alpha_i},y_\alpha\rangle|)$$

$$\leqq \mu_1+\lambda(1-\mu_1).$$

Hence $(1-\varepsilon)<\mu_1$ and therefore $\|x-x_\alpha\|\leqq 2\varepsilon$.

However we also want to give a direct proof of (a), which is almost trivial. Let $T: X\to Y$ be a continuous operator, $T\neq 0$ and $\varepsilon>0$. Find $\alpha\in I$ such that

$$\|Tx_\alpha\|\geqq\|T\|\cdot(1+\varepsilon\lambda)/(1+\varepsilon).$$

Define $\tilde{T}$ by

$$\tilde{T}x = Tx+\varepsilon\cdot\langle x, y_\alpha\rangle Tx_\alpha.$$

Then

$$\|\tilde{T}x_\alpha\|>\|T\|(1+\varepsilon\lambda)$$

while for $\beta\neq\alpha$

$$\|\tilde{T}x_\beta\|\leqq\|T\|\cdot(1+\lambda\cdot\varepsilon).$$

Hence $\tilde{T}$ attains its supremum at x_α and clearly $\|T-\tilde{T}\|\leqq\varepsilon\cdot\|T\|$. This finishes the (direct) proof of (a). □

We now turn to the duality between properties α and β. It will turn out that α is "predual" to a certain strengthening of β. We shall need some notation: If Y is a Banach space and X a subspace of Y^* we shall say that Y has *property β induced by X* if Y has property β and in addition to the requirements of Definition 1.2(b) we may assume that $x_\alpha\in X$ for all $\alpha\in I$. We now can formulate the duality result:

PROPOSITION 1.4. *Let X be a Banach space and $Y=X^*$. X has property α iff Y has property β induced by X.*

In particular, if Z is a reflexive Banach space then Z has property α (resp. β) iff Z^* has property β (resp. α).

PROOF. Just note that by the bipolar theorem the closed circled convex hull of $\{x_\alpha\}_{\alpha\in I}$ equals the unit ball of X iff its polar is the unit ball of Y, i.e. for $y\in Y$

$$\|y\|=\sup\{|\langle x_\alpha,y\rangle|:\alpha\in I\}.$$ □

A Banach space X is *uniformly convex* (see Day [2]) if

$$\delta(\varepsilon) = \inf\{1 - \|x + y\|/2 : \|x\| \leqq 1, \|y\| \leqq 1, \|x - y\| \geqq \varepsilon\}$$

is positive for all $0 < \varepsilon \leqq 2$. X is *super-reflexive* when it has an equivalent uniformly convex norm.

2.1. PROPOSITION (Partington [7]). *Let X be superreflexive. For every $K > 1$ there is*

(a) *a norm $|||\cdot|||$ on X, $\|\cdot\| \leqq |||\cdot||| \leqq K\cdot\|\cdot\|$ such that $(X, |||\cdot|||)$ has property* α,

(b) *a norm $|||\cdot|||$ on X, $\|\cdot\| \leqq |||\cdot||| \leqq K\cdot\|\cdot\|$ such that $(X, |||\cdot|||)$ has property* β.

REMARK 2.2. 2.1(b) has been proved in [7]. We shall give a very easy and intuitive proof of 2.1(a) (which is, however, based on the ideas of [7]); the duality result 1.5 gives immediately 2.1(b).

PROOF OF 2.1. We start with an observation used in [7]: If X is superreflexive it is arbitrarily nearly isometric to a uniformly convex space, for if $\|x\| \leqq p(x) \leqq C\|x\|$ and p is a uniformly convex norm on X then for $0 < r < 1$ the norm

$$q(x) = r\|x\| + (1 - r)p(x)$$

is uniformly convex and satisfies

$$\|x\| \leqq q(x) \leqq (r + C(1 - r))\|x\|.$$

We therefore may assume without loss of generality that $(X, \|\cdot\|)$ is uniformly convex.

Now let $0 < \varepsilon < 1$ be such that $K > (1 - \varepsilon)^{-1}$ and find a family $(x_\alpha)_{\alpha \in I}$ in X, $\|x_\alpha\| = 1$, which is maximal with respect to the condition that for $\alpha \neq \beta$

$$\|x_\alpha - x_\beta\| \geqq \varepsilon \quad \text{and} \quad \|x_\alpha + x_\beta\| \geqq \varepsilon.$$

Let B be the closed circled convex hull of $\{x_\alpha\}_{\alpha \in I}$ and $|||\cdot|||$ the Minkowski functional of B. Clearly

$$\|\cdot\| \leqq |||\cdot|||. \tag{1}$$

On the other hand note that for $x \in X$, $\|x\| = 1$ there is $\alpha(x) \in I$ such that

$$\|x - x_{\alpha(x)}\| < \varepsilon \quad \text{or} \quad \|x + x_{\alpha(x)}\| < \varepsilon. \tag{2}$$

Denote by $|||\cdot|||^*$ the dual Minkowski functional of $|||\cdot|||$ on X^* and by $\|\cdot\|^*$ the norm of X^*. For $y \in X^*$ we get in view of (2),

$$\begin{aligned}\|y\|^* &= \sup\{|\langle y, x\rangle| : \|x\| = 1\}\\ &\leqq \sup\{|\langle y, x_\alpha\rangle| + |\langle y, x\rangle| : \alpha \in I : \|x\| < \varepsilon\}\\ &= |||y|||^* + \varepsilon\|y\|^*.\end{aligned}$$

This implies that for $y \in X^*$

$$(1-\varepsilon)\|y\|^* \leqq |||y|||^*$$

and therefore for $x \in X$

$$K \cdot \|x\| \geqq |||x|||. \tag{3}$$

(1) and (3) give $\|\cdot\| \leqq |||\cdot||| \leqq K\|\cdot\|$. Now choose a family $\{y_\alpha\}_{\alpha\in I}$ in X^* subject to the condition

$$\|y_\alpha\| = \langle x_\alpha, y_\alpha\rangle = 1.$$

Then we have, for $\beta \neq \alpha$,

$$\langle x_\alpha, y_\beta\rangle < 1 - 2\delta(\varepsilon).$$

Indeed, if this were not the case, then

$$\langle (x_\alpha + x_\beta)/2, y_\beta\rangle \geqq 1 - \delta(\varepsilon).$$

Hence in particular

$$\|(x_\alpha + x_\beta)/2\| \geqq 1 - \delta(\varepsilon)$$

in contradiction to the inequality

$$\|x_\alpha - x_\beta\| \geqq \varepsilon.$$

Repeating the argument with x_α replaced by $-x_\alpha$ we obtain, for $\beta \neq \alpha$,

$$|\langle x_\alpha, y_\beta\rangle| < 1 - 2\delta(\varepsilon).$$

Hence $(X, |||\cdot|||)$ has property (α), which proves 2.1(a). It follows from 1.4 (or directly from the above construction) that $|||\cdot|||^*$ has property β. Also $\|\cdot\|^* \geqq |||\cdot|||^* \geqq K^{-1}\|\cdot\|^*$. Thus — interchanging the roles of X and X^* — we have also proved 2.1(b). □

3. Renorming c_0 to have property α

In this section we show that c_0 may be $(1+\varepsilon)$-equivalently renormed to have property α. We present this special case for several reasons: (1) The result is —

at least to the geometric intuition of the author — amazing; (2) in more general cases we do not obtain the constant $1+\varepsilon$; (3) the proof is technically simpler than the proof in more general circumstances, but it contains in a transparent way all the essential ingredients of the construction.

PROPOSITION 3.1. *For $K>1$ there is a norm $|||\cdot|||$ on c_0 with $\|\cdot\| \geqq |||\cdot||| \geqq K^{-1}\|\cdot\|$ and such that $(c_0, |||\cdot|||)$ has property α.*

In fact the constant λ appearing in Definition 1.2 may be chosen to be K^{-1}.

PROOF. Let $(\xi_n)_{n=1}^{\infty}$ be a dense sequence in the unit ball of c_0. Define x_n by

$$x_n(i) = \begin{cases} \xi_n(i) & \text{if } i \neq n, \\ K & \text{if } i = n. \end{cases}$$

Let B be the closed circled convex hull of $(x_n)_{n=1}^{\infty}$ and $|||\cdot|||$ the Minkowski functional of B. Clearly B is contained in the ball around zero of radius K, hence

$$|||\cdot||| \geqq K^{-1}\|\cdot\|.$$

On the other hand B contains the unit ball of c_0. Indeed let $x \in c_0$, $\|x\| \leqq 1$ and fix $\varepsilon > 0$. Let $m > 2(K+1)/\varepsilon$ and find $n_1 < \cdots < n_m$ such that

$$\|x - \xi_{n_j}\| < \varepsilon/2, \qquad j = 1, \cdots, m.$$

Let

$$\bar{x} = m^{-1} \sum_{j=1}^{m} x_{n_j};$$

then $\bar{x}$ is in B and $\|x - \bar{x}\| < \varepsilon$. Indeed if the coordinate i is different from $n_1, \cdots, n_m$, then

$$\|x(i) - \bar{x}(i)\| < \varepsilon/2.$$

For the coordinates n_j, where $1 \leqq j \leqq m$, we have the following estimate

$$|x(n_j) - \bar{x}(n_j)| \leqq m^{-1}\Bigl(|x(n_j) - x_{n_j}(n_j)| + \sum_{\substack{i=1 \\ i \neq j}}^{m} |x(n_j) - x_{n_i}(n_j)|\Bigr)$$

$$\leqq m^{-1}((K+1) + (m-1)\cdot\varepsilon/2) < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, B contains the unit ball of c_0, i.e. $\|\cdot\| \geqq |||\cdot|||$, which shows that $\|\cdot\|$ and $|||\cdot|||$ are K-equivalent.

Let $y_n = K^{-1}e_n$, where e_n denotes the n'th unit vector of l^1. Clearly

$$\langle x_n, y_n \rangle = 1$$

while for $m \neq n$

$$|\langle x_n, y_m \rangle| \leqq K^{-1}. \qquad \square$$

We can now harvest some corollaries; 3.2 and 3.3 answer open questions.

COROLLARY 3.2. *The Banach space c_0 equipped with the norm $|||\cdot|||$ constructed above has property* A *but fails the Bishop–Phelps property.*

PROOF. Indeed, it has been shown by J. Bourgain [1] that the Bishop–Phelps property is equivalent to the Radon–Nikodym property. $\square$

COROLLARY 3.3. *Property* A *is not invariant under isomorphism.*

PROOF. Indeed, it has been shown by J. Lindenstrauss [5] that c_0 fails property A. $\square$

COROLLARY 3.4. *l^1 may be $(1+\varepsilon)$-equivalently renormed to have property* B.

PROOF. Proposition 1.4 even implies that l^1 may be $(1+\varepsilon)$-equivalently renormed to have "property β induced by c_0".

4. Renorming general Banach spaces to have property α

In this section we try to "squeeze a maximal amount of juice out of the lemon", where the lemon is the proof of 3.1. Actually it would be fairer to say that the lemon is constituted by Partington's ideas.

THEOREM 4.1. *Let X be a Banach space and let the ordinal γ be the density character of X.*

Suppose that there is a family of sequences $((e_\alpha^n, f_\alpha^n)_{n=1}^\infty)_{\alpha<\gamma}$ with $e_\alpha^n \in X$, $f_\alpha^n \in X^*$, $\|e_\alpha^n\| = \|f_\alpha^n\| = 1$ and such that

(i) *there are constants $0 \leqq l < u \leqq 1$ such that for each (α, n)*

$$\langle e_\alpha^n, f_\alpha^n \rangle \geqq u$$

while for $(\alpha, n) \neq (\beta, m)$

$$|\langle e_\alpha^n, f_\beta^m \rangle| \leqq l;$$

(ii) *for each $\alpha < \gamma$ and $\varepsilon > 0$ there are scalars $(\lambda_n)_{n=1}^N$, $0 \leqq \lambda_n \leqq 1$, $\Sigma_{n=1}^N \lambda_n = 1$ such that for all choices of signs $\varepsilon_1, \cdots, \varepsilon_N$*

$$\left\| \sum_{n=1}^N \varepsilon_n \lambda_n e_\alpha^n \right\| < \varepsilon.$$

Then for $K > (u-l)^{-1}+1$ *there is a norm* $|||\cdot|||$ *on* X *with* $\|\cdot\| \geqq |||\cdot||| \geqq K^{-1}\|\cdot\|$ *such that* $(X, |||\cdot|||)$ *has property* α.

In fact, the constant λ *appearing in Definition* 1.2 *may be chosen to be* $((K-1)^{-1}+l)u^{-1}$.

PROOF. Let $(\xi_\alpha)_{\alpha<\gamma}$ be dense in the unit ball of X. For $\alpha<\gamma$ and $n \in \mathbf{N}$ define x_α^n and y_α^n in the following way:

If $\langle \xi_\alpha, f_\alpha^n\rangle \geqq 0$ let

$$x_\alpha^n = \xi_\alpha + (K-1)e_\alpha^n$$

and if $\langle \xi_\alpha, f_\alpha^n\rangle < 0$ let

$$x_\alpha^n = \xi_\alpha - (K-1)e_\alpha^n.$$

Define

$$y_\alpha^n = f_\alpha^n / \langle x_\alpha^n, f_\alpha^n\rangle.$$

Let B be the closed circled convex hull of $\{\{x_\alpha^n\}_{n=1}^\infty\}_{\alpha<\gamma}$ and $|||\cdot|||$ the Minkowski functional of B. Clearly B is contained in the ball around zero of radius K. Hence

$$|||\cdot||| \geqq K^{-1}\|\cdot\|.$$

On the other hand B contains the unit ball of X. Indeed let $x \in X, \|x\| \leqq 1$ and $\varepsilon > 0$. Find $\alpha < \gamma$ such that $\|x - \xi_\alpha\| < \varepsilon/2$. By hypothesis we may find positive scalars $(\lambda_n)_{n=1}^N$, $\Sigma\lambda_n = 1$ such that for all choices of signs $\varepsilon_1, \cdots, \varepsilon_N$

$$\left\| \sum_{n=1}^N \varepsilon_n \lambda_n e_\alpha^n \right\| < \varepsilon/2(K-1).$$

Hence

$$\left\| \xi_\alpha - \sum_{n=1}^N \lambda_n x_\alpha^n \right\| < \varepsilon/2$$

and therefore

$$\left\| x - \sum_{n=1}^N \lambda_n x_\alpha^n \right\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, B contains the unit ball of X, i.e.

$$\|\cdot\| \geqq |||\cdot|||.$$

Finally note that for $\alpha < \gamma$ and $n \in \mathbf{N}$

$$\langle x_\alpha^n, y_\alpha^n\rangle = 1.$$

On the other hand, if $(\alpha, n) \neq (\beta, m)$

$$\begin{aligned}
|\langle x^n_\alpha, y^m_\beta\rangle| &\leqq |\langle \xi_\alpha, y^m_\beta\rangle| + (K-1)|\langle e^n_\alpha, y^m_\beta\rangle| \\
&\leqq 1/(K-1)u + (K-1)(1/(K-1)u)\cdot l \\
&= (1/(K-1)+l)/u,
\end{aligned}$$

a constant strictly smaller than 1 by the choice of K. Hence $(X, |||\cdot|||)$ has property α. □

4.2. REMARK. The most obvious application of 4.1 is the following situation: Suppose there is a separable Banach space X and a biorthogonal sequence $(h_n, g_n)_{n=1}^\infty \subseteq X \times X^*$ with $\|h_n\| = 1$, $\|g_n\| \leqq M$ such that h_n tends to zero weakly. Indeed in this case we use a bijection from $\mathbf{N}$ to $\mathbf{N}\times\mathbf{N}$ to relabel $(h_n, g_n/\|g_n\|)_{n=1}^\infty$ as a double sequence $((e^n_m, f^n_m)_{n=1}^\infty)_{m<\omega}$. Then the conditions of 4.1 are satisfied with $u = M^{-1}$ and $l = 0$.

Note, however, that condition (ii) of 4.1 is weaker than requiring that for each $\alpha < \gamma$ the sequence $(e^n_\alpha)_{n=1}^\infty$ tends to zero weakly.

In order to take full advantage of this observation we need a folklore-type lemma, which is a straightforward consequence of Dvoretzky's theorem and the work of D. Amir and J. Lindenstrauss on WCG spaces:

4.3. LEMMA. *Let X be an infinite dimensional weakly compactly generated Banach space and let the ordinal γ be the density character of X and $\varepsilon > 0$.*

There is a family $((e^n_\alpha, f^n_\alpha)_{n=1}^\infty)_{\alpha<\gamma}$, $\|e^n_\alpha\| = \|f^n_\alpha\| = 1$,

$$\langle e^n_\alpha, f^n_\alpha\rangle \geqq 1/(2+\varepsilon) \qquad \forall(\alpha, n),$$

$$\langle e^n_\alpha, f^m_\beta\rangle = 0 \qquad \forall(\alpha, n) \neq (\beta, m),$$

and such that, for each $\alpha < \gamma$ and $k \in \mathbf{N}$, the elements $\{e^n_\alpha : 2^k < n \leqq 2^{k+1}\}$ are $(1+\varepsilon/k)$-equivalent to the unit vector basis of 2^k-dimensional l^2.

PROOF. Assume first that X is separable. Choose a normalized basic sequence $(x_i)_{i=1}^\infty$ with basis constant less than $1+\varepsilon/4$. Let $g_1 = x_1$. Now apply successively Dvoretzky's theorem (cf. [6]) to find an increasing sequence of integers $(i_k)_{k=1}^\infty$ with $i_1 = 1$ such that we may find linear combinations $g_{2^k+1}, \cdots, g_{2^k}$ of $x_{i_k+1}, \cdots, x_{i_{k+1}}$ which are $(1+\varepsilon/4k)$-equivalent to the unit vector basis of 2^k-dimensional l^2. Note that $(g_n)_{n=1}^\infty$ is then a basic sequence with basis constant less than $1+\varepsilon/2$. Hence we may find an orthogonal sequence $(h_n)_{n=1}^\infty$ in X^* with $\|h_n\| < 2+\varepsilon$.

Finally it is clear how to obtain the desired double-indexed sequence $((e_m^n, f_m^n)_{n=1}^{\infty})_{m<\omega}$. Just normalize the h_n's and relabel a suitably chosen sequence of the pairs $(g_n, h_n/\|h_n\|)$ such that each of the sequences $(e_m^n)_{n=1}^{\infty}$ contains almost isometric l^2-blocks in the prescribed way.

We now turn to the case, where X is not separable, i.e. $\gamma > \omega$. It follows from the work of Amir and Lindenstrauss (cf. [3], p. 140, lemma 5) that there is a "long sequence" of linear projections $(P_\alpha)_{\alpha<\gamma}$ such that

(i) $P_0 = 0$ and $\|P_\alpha\| = 1$ for $1 \leqq \alpha < \gamma$,

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ for $\beta < \alpha < \gamma$,

(iii) for every $\alpha < \gamma$ the range of $P_{\alpha+1} - P_\alpha$ is of infinite dimension.

Now find in every space $X_\alpha = (P_{\alpha+1} - P_\alpha)X$ as in the first part of the proof a basic sequence g_α^n with basis constant less than $1 + \varepsilon/2$ and such that for each k the finite sequence $\{g_\alpha^n\}_{n=2^k+1}^{2^{k+1}}$ is $(1+\varepsilon/k)$-isomorphic to the unit vector basis of 2^k-dimensional l^2. Let Z be the space spanned by $\{g_\alpha^n : n \in \mathbb{N}, \alpha < \gamma\}$. It is easily checked that the biorthogonal functionals $\tilde{h}_\alpha^n \in Z^*$ are of norm less than $2 + \varepsilon$ and may therefore be extended to $h_\alpha^n \in X^*$ with $\|h_\alpha^n\| < 2 + \varepsilon$.

Hence letting $e_\alpha^n = g_\alpha^n$ and $f_\alpha^n = h_\alpha^n / \|h_\alpha^n\|$ we have proved the lemma in the non-separable case too. □

QUESTION. Is it possible to replace $1/(2+\varepsilon)$ in Lemma 4.3 by $(1-\varepsilon)$?

4.4. THEOREM. *Let X be a* WCG *Banach space and $K > 3$. Then there is a norm $|||\cdot|||$ on X with $\|\cdot\| \geqq |||\cdot||| \geqq K^{-1}\|\cdot\|$ such that $(X, |||\cdot|||)$ has property α.*

PROOF. Choose a u so that $\frac{1}{2} > u > 1/(K-1)$ and apply Lemma 4.3 with $\varepsilon > 0$ so small that $1/(2+\varepsilon) > u$ and then apply Theorem 4.1 (where we take $l = 0$). □

Finally we shall give an example of a Banach space which is not WCG but to which Theorem 4.1 may be applied, namely l^∞. We could list some more examples but in general the above technique fails — as was kindly pointed out to us by the referee: Indeed, under the assumption $V = L$, S. Shelah has constructed a nonseparable Banach space X such that for every uncountable family $(x_\alpha)_{\alpha<\omega_1}$ and $\varepsilon > 0$ there are different $\alpha_0, \alpha_1, \cdots, \alpha_n$ and $\lambda_j \geqq 0$, $\Sigma_1^n \lambda_j = 1$ so that $\|x_{\alpha_0} - \Sigma_1^n \lambda_j x_{\alpha_j}\| < \varepsilon$. This property remains valid under any renorming; it follows easily from the definition that this space cannot be renormed satisfying property (α). We do not know whether it can be renormed to have property A.

4.5. PROPOSITION. *For $K > 2$ there is a norm $|||\cdot|||$ on l^∞ such that $\|\cdot\| \geqq |||\cdot||| \geqq K^{-1}\|\cdot\|$ and such that $(l^\infty, |||\cdot|||)$ has property α.*

PROOF. It is well known (and easily proved) that l^∞ contains $l^1(I)$ isometrically where I is a set of cardinality of the continuum.

Similar arguments as in Lemma 4.3 provide biorthogonal elements $\{(e_\alpha^n, f_\alpha^n) : n \in \mathbf{N}, \alpha < \Omega_c\}$ such that for each $\alpha < \Omega_c$ and $k \in \mathbf{N}$ the finite sequence $(e^n)_{n=2^k+1}^{2^{k+1}}$ is $(1+\varepsilon)$-isomorphic to 2^k-dimensional l^2 and such that $\|f_\alpha^n\| < 1 + \varepsilon$.

Hence we may apply 4.1 with $l = 0$ and $u = 1/(1+\varepsilon)$. □

To finish this section, we give a result on embedding:

4.6. THEOREM. *Every Banach space X may be* isometrically *and* 1-*complemented embedded into a Banach space Z with property α and having the same density character as X.*

PROOF. Let the ordinal γ be the density character of X and let I be the set

$$I = \{(\alpha, n) : \alpha < \gamma, n \in \mathbf{N}\}.$$

Let $Z = X \oplus c_0(I)$ with the norm

$$\|(x, w)\|_Z = \sup\{\|x\|_X, \|w\|_{c_0(I)}\}.$$

Denote i_X (resp. $i_{c_0(I)}$) the natural embedding of X (resp. $c_0(I)$) into Z and π_X the canonical projection from Z onto $i_X(X)$ along $c_0(I)$. Note that Z has the same density character as X.

Let e_α^n be the image of the (α, n)-th unit vector under $i_{c_0(I)}$ and $f_\alpha^n \in Z^*$ the biorthogonal functionals (i.e. f_α^n vanishes on $i_X(X)$ and on e_β^m for $(\beta, m) \neq (\alpha, n)$ while $\langle e_\alpha^n, f_\alpha^n \rangle = 1$).

Let $(\xi_\alpha)_{\alpha<\gamma}$ be dense in the unit ball of Z and define for $\alpha < \gamma$ and $n \in \mathbf{N}$

$$x_\alpha^n = \xi_\alpha + \lambda_\alpha^n e_\alpha^n$$

where the scalar λ_α^n is chosen such that

$$\langle x_\alpha^n, f_\alpha^n \rangle = 2.$$

Note that $\|x_\alpha^n\| = 2$. Define

$$y_\alpha^n = f_\alpha^n / 2.$$

Let $|||\cdot|||$ be the Minkowski-functional of the closed circled convex hull of $\{x_\alpha^n : \alpha < \gamma, n \in \mathbf{N}\}$. Clearly $B \cap i_X(X)$ equals the intersection of the unit ball of Z with $i_X(X)$ hence the embedding $i_X : X \to (Z, |||\cdot|||)$ is isometric. Also the projection $\pi_X : (Z, |||\cdot|||) \to i_X(X)$ has norm one. □

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